**Photon Field Quantization**

So in other contexts, the photon field is quantized somewhat differently. Often times one uses faux-Natural-Gaussian units, instead of Natural Lorentz-Heaviside units. And in condensed matter, one often presumes a finite volume space, V, rather than infinite, and so we use a discrete transform. So I’ll look at these possibilities.

**Coulomb gauge, using ‘Natural Lorentz-Heaviside’ units [discrete space]**

Basically going to reprise the discussion in the previous file, just making modifications for having a finite space, as we go. The photon field Hamiltonian is, in natural units (with c = ℏ = ε0 = 1 – what I’m calling Natural Lorentz Heaviside units I think – see Units file), given by:



where E(x) and B(x) are of course operators now (and | | means vector magnitude). In order to quantize the field, it is best to start with the action/Lagrangian as we’ve done before. So, recall/note that in these same units we have:



Everything can be written in terms of the space-time vector potential: Aα = (φ, **A**). And we have:



A problem we encounter is that we cannot impose canonical commutation relations on φ, because is missing. The underlying problem is that not all the Aα’s are independent. But we can fix the issue by choosing a gauge to eliminate the extraneous degrees of freedom, at least to the level necessary so that we can successfully quantize the theory.

One option is to go to the Coulomb gauge. We’ll recall from EM that this entails setting ∇·**A** = 0. But now it is also the case that in a source free region, which we do find ourselves in, it also follows that φ = 0 (can plainly see this from the discussion in EM file). We will need both of these conditions to formulate a self-consistent theory. The latter condition is easy to implement. This takes our L to:



But the former isn’t easy to implement in real space. We can do it in Fourier space though. Let’s make the transform,



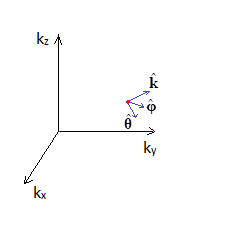
Then the constraint transforms to:



We can expand **A**(**k**,t) in a **k**-dependent basis using a resolution of identity:



which looks like this pictorally,



Note that these directions are the usual basis in spherical coordinates. And can be broken into Cartesian components (in k-space). So let **k** be given by



Then we have:



But it’s more typical to use a slightly different transverse basis.



where **ε**(**k**,λ) are just specified to be orthonormal unit vectors perpendicular to , and also switch signs when **k** → -**k**, just like they did in the phonon case:



Well, **θ** and **φ** do kind of do this themselves. Then we have:



This must be true for all x, and so must be true for all Ak. So we’ll set Ak = 0. Then,



Since **A** is Hermitian, we must have:



And so we must have:



If we plug this into L, we have:



Performing the x integral makes **k**´ = -**k**. And we get:



Now use the cross product property of the unit vectors, to write:



Probably better to rewrite it as:



Now let’s get the equation of motion for our Fourier space operators. Taking the functional derivative of the action



and setting to zero,



[this might be a little suspect as A and A† are correlated, but it gives the right result] Taking the dagger of this, we have a slightly nicer version:



The solution is:



And taking the inverse Fourier transform, we have:



Better to write this as:



We will recognize the coefficient of eiωt as the creation operator, and of e-iωt the annihilation operator, of excitations. And the excitations are indeed ωk = k (which would be ℏkc = pc if we restored the units). Defining,



Where we presume,



we can write:



(actually N depends on k, and should go inside the Σ) We’d like to work out these creation/annihilation operators and so we impose canonical commutation relations now. Noting,



These would (have) be(en):



(note this i,j subscript is now referring to x,y,z components) and now plugging in our FFE we have:



Let N = n√ωk, where n is yet to be determined. Then we have:



and



So now we have:



So we have finally,



When we plug this back into H, I’ll take on faith that we’ll get (see the next section for derivation in other units):



**Coulomb gauge, using ‘Natural Gaussian’ units [continuous space]**

Now let’s switch to Natural Gaussian units. These are defined via: ℏ = c = 4πε0 = 1. And we’re going to go back to continuous space. Going to just hit the major points. Now the EM Lagrangian is given by (can see EM folder for instance)



which reduces to in the Coulomb gauge:



The Fourier decomposition would give us:



Forming the action and minimizing, we’d get:



(ωk = ℏkc if all units restored) And taking the inverse Fourier transform, we have:



And we’d identify creation/annihilation operators:



with the typical commutation relations:



we can write:



(actually N depends on k, and should go inside the ∫d3k, but…) We’d like to work out these creation/annihilation operators and so we impose canonical commutation relations now. Noting,



These would (have) be(en):



(note this i,j subscript is now referring to x,y,z components) and now plugging in our FFE we have:



Let N = n√ωk, where n is yet to be determined. Then we have:



and so,



So now we have:



So we have finally,



Let’s put this into H. Going back to L:



and constructing H,



Then filling in our A, at t = 0, I presume we’ll get:



**Coulomb gauge, using ‘Natural Gaussian’ units [discrete space]**

So, recall/note that in these units we have (setting ℏ = c = 4πε0 = 1) (can see EM folder for instance)



which reduces to in the Coulomb gauge:



I’ll just brush through the details. The (finite) Fourier decomposition would give us:



Forming the action and minimizing, we’d get:



(ωk = ℏkc if all units restored) And taking the inverse Fourier transform, we have:



And we’d identify creation/annihilation operators:



with the typical commutation relations:



we can write:



(actually N depends on k, and should go inside the Σ) We’d like to work out these creation/annihilation operators and so we impose canonical commutation relations now. Noting,



These would (have) be(en):



(note this i,j subscript is now referring to x,y,z components) and now plugging in our FFE we have:



Let N = n√ωk, where n is yet to be determined. Then we have:



and



So now we have:



So we have finally,



Let’s put this into H. Going back to L:



and constructing H,



Then filling in our A, at t = 0,



and,



Now recalling,



we have:



and so we have:



**Alternate Phase Conventions (i.e. Mahan in Cond Mat)**

In condensed matter physics, I’ve seen other expressions for the free field expansion, which can more or less be obtained by a phase transformation on the a’s. I’ll call this the ‘CMT’ phase convention, as opposed to the ‘QFT’ phase convention we have been using. Define b = -ia. Then b and b† would still obey the appropriate canonical commutation relations. Then we can say, for instance,



and further, we could switch k→-k in the b† sum,



Only thing is, Mahan for instance doesn’t have the i out front. But this would seem to make A not Hermitian. Our present definition does make it Hermitian,



but if we took that pre-factor i away, it’d be anti-Hermitian. So think he’s wrong.